

# Non-Archimedean regulator maps and special values of $L$ -functions.

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## Abstract

We define an analogue of the ‘Real’ Deligne cohomology group at a prime of semi-stable or good reduction of a variety. We also define regulator maps to this group and formulate a conjecture about the image. This allows us to formulate a non-Archimedean version of Beilinson’s Hodge- $\mathcal{D}$ -conjecture and  $S$ -integral and function field versions of Beilinson’s global conjectures as well as a precise special value conjecture in the function field case. Finally we give a few examples where these conjectures are known to be true.

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## 1 Introduction

Let  $X$  be a smooth projective variety over  $\mathbb{Q}$ . Beilinson [Be] formulated conjectures relating special values of  $L$ -functions to the  $K$ -theory of such varieties in terms of the ‘Real’ Deligne cohomology of the varieties. Roughly speaking, he constructed a regulator map from a higher Chow group of the variety to a real vector space, the Real Deligne cohomology, such that the image gives a  $\mathbb{Q}$ -structure. The dimension of this vector space is the order of pole of the Archimedean<sup>1</sup> factor of a cohomological  $L$ -function of the variety. Finally the real vector space has another lattice structure induced by the Betti and de Rham cohomology groups and the determinant of the the change of basis matrix is related to a special value of this  $L$ -function.

In this paper we define an analogue of the Deligne cohomology group for a finite prime of good or strict semi-stable reduction. We show there is a regulator map from the higher Chow group to this Deligne cohomology group and show that it has similar properties for known or conjectural reasons.

The original aim of this paper was to formulate a version of Beilinson’s conjectures in the case of varieties over function fields of characteristic  $p$  so as to put the results of [Ko], [Pàl] and [C-S] in a general framework. Since in this case all the primes are finite we do have such a formulation. Further, we can formulate an  $S$ -integral version of the Beilinson conjectures. Finally, we can also formulate a precise special value conjecture in the spirit of Bloch-Kato [B-K].

While the definition of the regulator map is considerably simpler than the Archimedean case, in some cases there are very similar formulas. The link between the two seems to be through the theory of  $p$ -adic uniformization.

It is generally believed that a variety should be considered to have totally degenerate reduction at an Archimedean place. In particular it has semi-stable reduction, so the usual conjectures are just the statements in the case of an Archimedean prime.

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<sup>1</sup>There seems to be some discrepancy in the literature as to the spelling of ‘Archimedean’ - an alternative is ‘Archimedian’. However, a search on Google revealed that the former is more popular, so that is what we have used.

In the final section we show some examples where these conjectures are known to be true. To a certain extent we re-interpret known results in our terms, so we may be guilty of putting old wine in new bottles. Further, barring the function field case, the conjectures formulated here were perhaps implicitly, if not explicitly, known to the experts - though as far as we are aware they have not appeared in print - at least from this point of view.

Finally, the correct context for the conjectures should be motives, but we have chosen to describe them in terms of varieties for ‘simplicity’.

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## 2 The Archimedean Case

The usual Real Deligne cohomology  $H_{\mathcal{D}}^q(X/\mathbb{R}, \mathbb{R}(q-a))$ , with  $q > 2a+1$  has the following properties.

- 1. It is a finite dimensional real vector space with

$$\dim_{\mathbb{R}} H_{\mathcal{D}}^q(X/\mathbb{R}, \mathbb{R}(q-a)) = -ord_{s=a} L_{\infty}(H^{q-1}(X), s)$$

the L-factor at the Archimedean place.

- 2. There is a regulator map  $r_{\mathcal{D}} : CH^{q-a}(X, q-2a) \otimes \mathbb{Q} \longrightarrow H_{\mathcal{D}}^q(X/\mathbb{R}, \mathbb{R}(q-a))$
- 3. There is a  $\mathbb{Q}$  structure on this real vector space induced by the Betti cohomology group  $H_B^{q-1}(X(\mathbb{C}), \mathbb{Q})$  and piece of the de Rham cohomology group  $F^{q-a}H_{dR}^{q-1}(X/\mathbb{R})$ .
- 4. The image of the regulator map is conjecturally another  $\mathbb{Q}$ -lattice in the real vector space.
- 5. Assuming 4 one can compute the determinant of the change of basis matrix with respect to these two lattices. Let  $c_{\infty}(X, q, a)$  be that number. Then conjecturally

$$L^*(X, a) \sim_{\mathbb{Q}^*} c_{\infty}(X, q, a)$$

where  $L^*(X, a)$  denotes the first non-zero value of the Laurent expansion of  $L(X, s)$  at  $s = a$ .

We will define a  $\mathbb{Q}$  vector space for a prime  $p$  of semi-stable or good reduction which has property 1 owing to the work of Consani, [Co]. We will then define a regulator map to this vector space and will speculate on analogues of properties 3,4 and 5.

When  $q - 2a = 1$  ( corresponding to  $K_1$ ) the conjecture has to be slightly modified - one has to add a term corresponding to the group  $B^a(X) = CH^a(X)/CH_{hom}^a(X)$  and a similar statement holds. Our formulation takes this in to account as well.

Beilinson [Be] formulated his conjectures in terms of graded pieces of the  $K$ -theory of the varieties but we have chosen to formulate it in terms of the higher Chow groups. If one is not interested in a precise special value conjecture it does not make a difference but for the exact value it could. However, in the cases for which we have examples it does not. There is no particularly good reason for our choice. All of the formalism goes through for any candidate for motivic cohomology, so if the higher Chow groups fail, it is plausible that some other candidate could succeed.

### 3 Preliminaries

Let  $X$  be a smooth proper variety over a field  $K$  and  $\Lambda$  a discrete valuation ring with closed point  $v$  and generic point  $\eta$ .

By a model  $\mathcal{X}$  of  $X$  we mean a flat proper scheme  $\mathcal{X} \rightarrow \text{Spec}(\Lambda)$  together with an isomorphism of the generic fibre  $X_\eta$  with  $X$ . Let  $Y$  be the special fibre  $X \times \text{Spec}(k(v))$ . We will always also make the assumption that the model is strictly semi-stable, which means that it is a regular model and the fibre  $Y$  is a divisor with normal crossings, the components have multiplicity one and they intersect transversally.

We have the following picture

$$\begin{array}{ccccc} Y & \xrightarrow{i} & \mathcal{X} & \xleftarrow{j} & \bar{X} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec}(k(v)) & \longrightarrow & \text{Spec}(\Lambda) & \longleftarrow & \text{Spec}(\bar{\eta}) \end{array}$$

### 4 Consani's Double Complex

In [Co], Consani defined a double complex of Chow groups of the components of the special fibre with a monodromy operator  $N$ , following the work of Steenbrink [St] and Bloch-Gillet-Soulé [BGS]. We need to use this complex to define the Deligne cohomology in the case of strict semistable reduction. To define it we need some preliminaries.

Let  $Y = \coprod_{i=1}^t Y_i$  be the special fibre of dim  $n$  with  $Y_i$  its irreducible components. For  $I \subset \{1, 2, \dots, t\}$ , define

$$Y_I = \cap_{i \in I} Y_i$$

Let  $r = |I|$  denote the cardinality of  $I$ . Define

$$Y^{(r)} := \begin{cases} \mathcal{X} & \text{if } r = 0 \\ \coprod_{|I|=r} Y_I & \text{if } 1 \leq r \leq n \\ \emptyset & \text{if } r > n \end{cases}$$

For  $u$  and  $t$  with  $1 \leq u \leq t < r$  define the map

$$\delta(u) : Y^{(t+1)} \rightarrow Y^{(t)}$$

as follows. Let  $I = (i_1, i_2, \dots, i_t)$  with  $i_1 < i_2 < \dots < i_t$ . Let  $J = I - \{i_u\}$ . There is an embedding

$$Y_I \rightarrow Y_J$$

and this induce maps on the cohomology and homology of these varieties. Let  $\delta(u)^*$  and  $\delta(u)_*$  denote the corresponding maps. They further induce the Gysin and restriction maps on the cohomology and homology as follows.

Define

$$\gamma := \sum_{u=1}^r (-1)^{u-1} \delta(u)_*$$

and

$$\rho := \sum_{u=1}^r (-1)^{u-1} \delta(u)^*$$

These maps have the properties that

- $\gamma^2 = 0$
- $\rho^2 = 0$
- $\gamma \cdot \rho + \rho \cdot \gamma = 0$

Let  $i, j, k \in \mathbb{Z}$ . Define, following [Co](3.1)

$$K^{i,j,k} := \begin{cases} CH^{\frac{i+j-2k+n}{2}}(Y^{(2k-i+1)}) \otimes \mathbb{Q} & \text{if } k \geq \max(0, i) \\ 0 & \text{otherwise} \end{cases}$$

and let

$$K^{i,j} = \oplus_k K^{i,j,k} \text{ and } K^n = \oplus_{i+j=n} K^{i,j}$$

The maps  $\rho$  and  $\gamma$  induce differentials

$$\begin{aligned} \partial' : K^{i,j,k} &\rightarrow K^{i+1,j+1,k+1} & \partial'(a) &= \rho(a) \\ \partial'' : K^{i,j,k} &\rightarrow K^{i+1,j+1,k} & \partial''(a) &= -\gamma(a) \end{aligned}$$

Further define

$$N : K^{i,j,k} \rightarrow K^{i+2,j,k+1}(-1) \quad N(a) = a$$

Let  $\partial = \partial' + \partial''$  on  $K^{i,j}$ . From the definition we have  $[\partial, N] = 0$  and  $\partial^2 = 0$ .

Let  $Cone(N) : K^* \rightarrow K^*$  be the complex  $K^* \oplus K^*[-1]$  with differential

$$D(a, b) = (\partial(a), N(a) - \partial(b))$$

Consani [Co][Prop 3.4] shows that this cone complex is quasi-isomorphic to a complex of Chow groups of the fibre:

**Proposition 4.1 (Consani).** *Let  $*$  be a fixed integer. The complex, for  $q \in \mathbb{Z}$ ,*

$$Cone(N : K^{q-2*, q-n} \rightarrow K^{q-2*+2, q-n})$$

*is quasi-isomorphic to the following complex*

$$C^q(*) = \begin{cases} CH^{q-*}(Y^{(2*-q)}) & \text{if } q \leq * - 1 \\ CH^*(Y^{(q-2*)}, \mathbb{Q}_\ell) & \text{if } q \geq * \end{cases}$$

The differential  $d_C$  is given by

$$d_C(a) = \begin{cases} d''(a) & \text{if } q < * - 1 \\ -i^* i_*(a) & \text{if } q = * - 1 \\ d'(a) & \text{if } q \geq * \end{cases}$$

## 5 The ‘Deligne cohomology’ at a finite prime

Assume now that the residue field  $k(v)$  is finite <sup>2</sup>. We define the  $v$ -adic Deligne Cohomology group to be

$$H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q-a)) := \begin{cases} CH^{q-a-1}(Y, q-2a-1) \otimes \mathbb{Q} & \text{if } q-2a > 1 \\ \frac{Ker(i^*i_*: CH_{n-a}(Y^{(1)}) \rightarrow CH_{n-a}(Y^{(1)}))}{Im(\gamma: CH_{n-a}(Y^{(2)}) \rightarrow CH_{n-a}(Y^{(1)}))} \otimes \mathbb{Q} & \text{if } q-2a = 1 \end{cases}$$

Here  $n$  is the dimension of  $Y$ . This is a  $\mathbb{Q}$  vector space which we will show has the expected properties assuming certain conjectures. Note that if  $Y$  is non-singular and  $q-2a > 1$  the Parshin-Soulé conjecture asserts that the higher Chow group is finite, hence this space is 0. When  $q-2a = 1$  the group is  $CH^a(Y) \otimes \mathbb{Q}$ .

**Remark 5.1.** *Consani shows that if one uses a certain complex of differential forms to define  $K^{i,j,k}$  and performs the same calculations, one ends up with the Real Deligne cohomology as the graded pieces.*

## 6 Properties of the Deligne Cohomology

### 6.1 Dimension:

The usual Real Deligne cohomology has the property that its dimension is the order of the pole of the Archimedean part of the  $L$ -function at a certain point on the left of the critical point. Here we have a similar property. Let  $F^*$  be the geometric Frobenius and  $N(v)$  the number of elements of  $k(v)$ . The local  $L$ -factor of the  $(q-1)^{st}$  cohomology group is then

$$L_v(X, s) = (\det(I - F^* N(v)^{-s} | H^{q-1}(\bar{X}, \mathbb{Q}_\ell)^I))^{-1}$$

**Theorem 6.1 (Consani).** *Let  $v$  be a place of semistable reduction. Assuming the weight-monodromy conjecture, the Tate conjecture for the components and the injectivity of the cycle class map on the components  $Y_I$ , Parshin-Soulé conjecture and that  $F^*$  acts semisimply on  $H^*(\bar{X}, \mathbb{Q}_\ell)^I$ . we have*

$$\dim_{\mathbb{Q}}(H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q-2a))) = -ord_{s=a} L_v(X, s) := d_v$$

*Proof.* [Co], Thm 3.5 □

**Remark 6.2.** *Since the  $L$ -factor at a prime of good reduction does not have a pole at  $s = a$  when  $q-2a > 1$ , the Parshin-Soulé conjecture can be interpreted as the statement that the  $v$ -adic Deligne cohomology has the correct dimension, namely 0, even at a prime of good reduction.*

**Remark 6.3.** *In the function fields setting and more recently, in the setting of  $p$ -adically uniformized varieties, the weight-monodromy conjecture is a theorem [De], [It]. Further, in the  $p$ -adically uniformized case, the variety is totally degenerate so the Tate conjecture for the components is trivial, so assuming semi-simplicity of the action of the Frobenius and injectivity of the cycle class map on the components, Consani’s theorem holds in this case.*

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<sup>2</sup>This is necessary to define the  $L$ -series and is not strictly necessary at this point.

## 6.2 Regulator maps

To define the regulator map we use the localization sequence [Bl]. It is as follows. If  $X, \mathcal{X}$  and  $Y$  are as before, and  $q, a \in \mathbb{Z}, q - 2a > 0$  we have

$$\begin{aligned} \cdots \rightarrow CH^{q-a}(\mathcal{X}, q - 2a) \rightarrow CH^{q-a}(X, q - 2a) \xrightarrow{\partial} CH^{q-a-1}(Y, q - 2a - 1) \\ \rightarrow CH^{q-a}(\mathcal{X}, q - 2a - 1) \rightarrow CH^{q-a}(X, q - 2a - 1) \rightarrow \cdots \end{aligned}$$

The usual regulator map should appear as the boundary map in the ‘arithmetic’ localization sequence.

$$\cdots \rightarrow \widehat{CH}^{q-a}(\mathcal{X}, q - 2a) \rightarrow CH^{q-a}(\mathcal{X}, q - 2a) \xrightarrow{r_{\mathcal{D}}} H_{\mathcal{D}}^{2a+1}(X/\mathbb{R}, \mathbb{R}(a + 1)) \rightarrow \cdots$$

As far as we are aware, higher arithmetic Chow groups have not been defined in general, but this is known in the case when  $q - 2a = 1$ .

In the finite prime case, there are two cases that have to be considered.

### 6.2.1 Case 1: $q - 2a > 1$

We define the *v-adic regulator map* to be the map  $\partial$ .

$$r_{\mathcal{D},v} : CH^{q-a}(X, q - 2a) \xrightarrow{\partial} H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q - a))$$

In analogy with the Beilinson conjectures, we have the following conjecture

**CONJECTURE A1:** The image

$$Im(r_{\mathcal{D},v}(CH^{q-a}(X, q - 2a))) \subset H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q - a))$$

is a full sub-lattice.

### 6.2.2 Case 2: $q - 2a = 1$

In this case the Chow groups of the components of the special fibre are not torsion, so the conjecture has to be slightly modified. However, we get a conjecture which is non trivial even in the case of good reduction.

One has a presentation of the Chow group

$$CH^*(Y) = Coker(\gamma : CH_{n-*}(Y^{(2)}) \rightarrow CH_{n-*}(Y^{(1)}))$$

Recall that  $i : Y \hookrightarrow \mathcal{X}$  is the inclusion map of the special fibre into the model. From the exactness of the localization sequence, the image of  $\partial$  lies in the kernel of

$$i_* : CH_{n-*}(Y) \rightarrow CH_{n-*}(\mathcal{X}) = CH^{*-1}(\mathcal{X})$$

In particular, it lies in the kernel of  $i^*i_* : CH_{n-*}(Y^{(1)}) \rightarrow CH_{n-*}(Y^{(1)})$ .

We define the *v-adic regulator map* as before as map  $\partial$

$$r_{\mathcal{D},v} : CH^{a+1}(X, 1) \xrightarrow{\partial} H_{\mathcal{D}}^{2a+1}(X/v, \mathbb{Q}(a + 1))$$

and we have the following conjecture.

**CONJECTURE A2:** The map

$$Im(r_{\mathcal{D},v}(CH^{a+1}(X, 1))) \subset H_{\mathcal{D}}^{2a+1}(X/v, \mathbb{Q}(a + 1))$$

is a full sublattice.

### 6.2.3 The map $z_v^a$

We have another map

$$z_v^a : CH^a(X) \longrightarrow H_{\mathcal{D}}^{2a+1}(X/v, \mathbb{Q}(a+1))$$

defined as follows. The map  $j^* : CH^a(\mathcal{X}) \rightarrow CH^a(X)$  is surjective and we have the map  $i^* : CH^a(\mathcal{X}) \rightarrow CH^a(Y)$ . This induces a map  $i^* \circ (j^*)^{-1} : CH^a(X) \rightarrow CH^a(Y)$  which is well defined up to an element of the image of  $i_*^* : CH^{a-1}(Y) \rightarrow CH^a(Y)$ . So we have a well defined map

$$\xi_v^a : CH^a(X) \rightarrow \frac{Ker(\rho : CH^a(Y^{(1)}) \rightarrow CH^a(Y^{(2)}))}{Im(i_*^*)} \otimes \mathbb{Q}$$

There is always a morphism  $\tau$  from the group on the right to the Deligne cohomology induced by the cap product  $\cap[Y]$  [BGS], pg. 454., and we define

$$z_v^a := \tau \circ \xi_v^a$$

In some instances, for example, if  $Y$  is smooth [BGS][Prop 2.3.3], the map  $\tau$  is known to be an isomorphism. In several other instances [BGS][Section 6], the map is known to be an isomorphism after going to cohomology.

### 6.2.4 The Archimedean case

The Archimedean case of **Conjecture A** is Beilinson's Hodge- $\mathcal{D}$ -conjecture. However, this is known to be false in general [M-S]. It was suggested by N.Fakhruddin [Fa] that perhaps similar methods can be used to show that **Conjecture A** is false over p-adic fields, but may still hold over dvr's whose quotient field is contained in the algebraic closure of a number field. As in all the examples we have the conjecture is true, we have left it as it is. However, in all our examples the Hodge- $\mathcal{D}$ -conjecture is also known.

The Archimedean version of the map  $z^a$  comes from the usual cycle class map to Betti cohomology and the long exact sequence relating the Betti, de Rham and Deligne cohomologies.

## 7 The $S$ -integral Beilinson conjecture

The conjectures above can be combined with the Hasse-Weil conjecture to formulate an  $S$ -integral version of the Beilinson conjectures. Let  $X$  be a variety defined over a number field  $K$  which has at worst semi-stable reduction at all places ( this restriction may not be so important but at the moment it is necessary ). Let  $\mathcal{O}_K$  be the ring of integers of  $K$  and let  $\infty$  denote the set of Archimedean places. Let  $L_v(X, s)$  be a cohomological  $L$ -function at the place  $v$  for the  $(q-1)^{st}$  cohomology group as defined above, where for  $v$  Archimedean it is defined as in [R-S-S] pg 4. Define the completed  $L$ -function  $\Lambda(X, s)$  as follows:

$$\Lambda(X, s) = A^{s/2} \prod_v L_v(X, s)$$

where  $A$  is a generalized conductor.

The **Standard Conjectures** of Grothendieck, generalizing those of Hasse-Weil and Serre, asserts that this function has a meromorphic continuation to the entire complex plane

$$\Lambda(X, s) = \pm \Lambda(X, q-s)$$

For a set of places  $S$  containing the Archimedean places we define the  $S$ -integral  $L$ -function as follows:

$$L_S(X, S) = \prod_{s \notin S} L(X, s)$$

so

$$\Lambda(X, s) = L_S(X, s) \prod_{v \in S} L_v(X, s)$$

The the above conjectures can be combined with the usual Beilinson conjectures to give the following  $S$ -integral Beilinson conjectures: Let  $c_\infty(X, q, a)$  be the number appearing the usual Beilinson conjectures, coming from isomorphism of the two  $\mathbb{Q}$ -structures and for a finite set of places  $S$  containing the Archimedean places, let  $\mathcal{X}_S$  be a model over  $\mathcal{O}_K[S]$  with at worst semistable reduction.

**CONJECTURE B1:** Let  $S$  be a finite set of places containing all the Archimedean places and  $q - 2a > 1$ . Then

- $ord_{s=a} L_S(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, q - 2a) \otimes \mathbb{Q}$
- The map

$$\oplus_{v \in S \setminus \infty} r_{\mathcal{D}, v} \otimes \mathbb{Q} \bigoplus \oplus_{v | \infty} r_{\mathcal{D}, v} \otimes \mathbb{R} : CH^{q-a}(\mathcal{X}_S, q - 2a) \longrightarrow \bigoplus_{v \in S} H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q - a))$$

is an isomorphism.

- $L_S^*(X, a) \sim_{\mathbb{Q}^*} c_\infty(X, q, a) \prod_{v \in S \setminus \infty} (\log(N(v)))^{d_v}$

**CONJECTURE B2:** Let  $S$  be a finite set of places containing the Archimedean places and  $q - 2a = 1$ . Let  $B^a(X)$  denote the group  $(CH^a(X)/CH_{hom}^a(X))$  and  $z^a$  the cycle class map to Deligne cohomology induced by the cycle class map to  $H_B^{2a}(X, \mathbb{R}(a))^{(-1)^a}$

Then

- $ord_{s=a} L_S(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, 1) \otimes \mathbb{Q}$
- $ord_{s=a+1} L_S(X, s) = -\dim_{\mathbb{Q}} B^a(X)$  ( Tate's Conjecture)
- The map

$$\oplus_{v \in S \setminus \infty} r_{\mathcal{D}, v} \otimes \mathbb{Q} \oplus \oplus_{v | \infty} r_{\mathcal{D}, v} \otimes \mathbb{R} \oplus z^a : CH^{q-a}(\mathcal{X}_S, 1) \oplus B^a(X) \otimes \mathbb{Q} \rightarrow \oplus_{v \in S} H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q - a))$$

is an isomorphism.

- $L_S^*(X, a) \sim_{\mathbb{Q}^*} c_\infty(X, q, a) \prod_{v \in S \setminus \infty} (\log(N(v)))^{d_v}$

## 8 Function Fields

In the case of function fields, we can formulate a conjecture along the lines of **Conjecture B** using the fact that the primes at  $\infty$  are just finite primes.

Let  $\mathbb{F}_q$  be the finite field of order  $q$  and  $K$  a function field over  $\mathbb{F}_q$  of transcendence degree 1. Let  $S$  be a non empty finite set of primes of  $K$ . Then there exists an element  $x$  in  $K$  whose poles are



precisely the elements of  $S$ . Let  $\mathcal{O}_S$  denote the integral closure of  $\mathbb{F}_q[x]$  in  $K$ . The prime  $\infty$  is the prime  $1/x$  of  $\mathbb{F}_q[x]$  and the set  $S$  is the set of primes lying over  $\infty$ . Let  $\mathcal{X}_S$  be a model of  $X$  over  $\mathcal{O}_S$  with at worst semi-stable reduction.

**CONJECTURE B1FF:** Let  $S$  be a finite set of places and  $\mathcal{X}_S$  as above and  $q - 2a > 1$  Then

- $ord_{s=a} L_S(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, q - 2a) \otimes \mathbb{Q}$

- The map

$$\bigoplus_{v \in S} r_{\mathcal{D}, v} \otimes \mathbb{Q} : CH^{q-a}(\mathcal{X}_S, q - 2a) \rightarrow \bigoplus_{v \in S} H_{\mathcal{D}}^q(X/v, \mathbb{Q}(q - a))$$

is an isomorphism.

- $L_S^*(X, a) \sim_{\mathbb{Q}^*} \prod_{v \in S} (\log(N(v)))^{d_v}$

In the case  $q - 2a = 1$ , let  $B^a(X)$  denote the group  $(CH^a(X)/CH_{hom}^a(X))$  and let  $z_S^a$  denote the map

$$z_S^a := \bigoplus_{s \in S} z_v^a : B^a(X) \rightarrow \bigoplus_{s \in S} H_{\mathcal{D}}^{2a+1}(X/v, \mathbb{Q}(a + 1))$$

induced by the restriction maps  $CH^a(X) \rightarrow CH^a(Y_v)$ .

**CONJECTURE B2FF:** If  $q - 2a = 1$ , Let  $S$  be a finite set of places and  $\mathcal{X}_S$  as above. Then

- $ord_{s=a} L_S(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(\mathcal{X}_S, 1) \otimes \mathbb{Q}$

- $ord_{s=a+1} L_S(X, s) = -\dim_{\mathbb{Q}} B^a(X) \otimes \mathbb{Q}$  ( Tate's Conjecture )

- The map

$$\bigoplus_{v \in S} r_{\mathcal{D}, v} \otimes \mathbb{Q} \oplus z_S^a \otimes \mathbb{Q} : CH^{a+1}(\mathcal{X}_S, 1) \oplus B^a(X) \rightarrow \bigoplus_{v \in S} H_{\mathcal{D}}^{2a+1}(X/v, \mathbb{Q}(a + 1))$$

is an isomorphism.

- $L_S^*(X, a) \sim_{\mathbb{Q}^*} \log(q)^{\dim(CH^{a+1}(\mathcal{X}_S, 1))}$

Notice that the only transcendental term is a power of  $\log(q)$  which comes from the residue of the local  $L$ -function.

## 9 A Special value conjecture

The considerations of the previous sections allow us to formulate a special value conjecture in the function field case. Perhaps this is the same as the Bloch-Kato conjecture [B-K] but we have not verified that. Formulating the conjecture here is a little easier as in the number field case, Beilinson's regulator map is only defined up to  $\mathbb{Q}^*$ .

## 9.1 $\mathbb{Z}$ -structures

To conjecture an expression for the special value, we use the fact that the Deligne cohomology at a finite prime has two  $\mathbb{Z}$  structures, one natural and the other conjectural. The first  $\mathbb{Z}$  structure comes from the integral Chow group  $CH^{q-a-1}(Y_v, q-2a-1)$ . **Conjecture A** asserts that the image of the regulator map gives a second  $\mathbb{Z}$  structure, which is a subgroup of the integral Chow group.

From the earlier conjectures when  $q-2a > 1$ , the map

$$CH^{q-a}(X, q-2a) \otimes \mathbb{Q} \xrightarrow{\oplus r_{\mathcal{D},v}} \bigoplus_v CH^{q-a-1}(Y_v, q-2a-1) \otimes \mathbb{Q}$$

is an isomorphism. Similarly, when  $q-2a = 1$ , the map

$$CH^{a+1}(X, 1) \otimes \mathbb{Q} \oplus B^a(X) \otimes \mathbb{Q} \xrightarrow{\oplus r_{\mathcal{D},v} \oplus z_v^a} \bigoplus_v CH^a(Y_v) \otimes \mathbb{Q}$$

is an isomorphism. However, the maps are defined integrally, so the kernel and cokernel are torsion. We conjecture that they are actually finite. Let  $b(X, q, a)$  denote the order of the kernel and  $c(X, q, a)$  denote the order of the cokernel.

## 9.2 A special value conjecture

We then have the following conjecture:

### CONJECTURE CFF:

- C1: If  $q-2a > 1$  then

$$\text{ord}_{s=a} \Lambda(X, s) = \dim_{\mathbb{Q}} CH^{q-a}(X, q-2a) \otimes \mathbb{Q}$$

- C2: If  $q-2a = 1$ , the space  $CH^{q-a}(X, 1) \otimes \mathbb{Q}$  is **not** finite dimensional, but

$$\text{ord}_{s=a} \Lambda(X, s) = -\dim_{\mathbb{Q}} B^a(X) \otimes \mathbb{Q}$$

Further,

$$\Lambda^*(X, a) = \pm \frac{c(X, q, a)}{b(X, q, a)} \log(q)^{\text{ord}_{s=a} \Lambda(X, s)}$$

**Remark 9.1.** One can also formulate an exact value conjecture in the  $S$ -integral case and this can be viewed as the conjecture when  $S$  is the set of all primes.

## 10 Examples

### 10.1 $K_1$ of fields

This is the only case where all the conjectures are known. Here  $q = 1$  and  $a = 0$ . Let  $K$  be a number field and  $\Lambda$  the completion of  $\mathcal{O}_K$  at a finite place  $v$ . Let  $\eta$  denote the generic point of  $\text{Spec}(\Lambda)$ .

### 10.1.1 Local Case - Conjecture A:

As  $q - 2a = 1$ , we are in case 2. Here the  $L$ -function is

$$L_v(\text{Spec}(K), s) := L_v(H^0(\text{Spec}(\Lambda), \mathbb{Q}_\ell(0)), s) = \frac{1}{1 - N(v)^{-s}}$$

which has a pole at  $s = 0$ . The conjecture A2 then asserts that the rank of the image of the regulator map is 1.

The localization sequence gives

$$\cdots \rightarrow CH^1(\text{Spec}(\Lambda), 1) \rightarrow CH^1(\eta, 1) \rightarrow CH^0(v) \rightarrow CH^1(\text{Spec}(\Lambda)) \rightarrow CH^1(\eta) \rightarrow 0$$

Since  $\Lambda$  is a principal ideal domain,  $CH^1(\Lambda) = 0$ , so the regulator map is surjective and the image has of the regulator map is rank 1. So this is a case when the conjecture is true. Notice this is a case in which the variety has good reduction.

The statement that the map

$$\mathbb{R}^* \xrightarrow{\log ||} \mathbb{R}$$

is surjective, and similarly for  $\mathbb{C}^*$ , can be viewed as an *Archimedean local case*.

### 10.1.2 Global Case - Conjectures B and C :

Let

$$\zeta_{K,S}(s) = \prod_{v \notin S} L_v(\text{Spec}(K), s)$$

**Conjecture B2** is the usual  $S$ -unit theorem of Dirichlet. The fact that the regulator map is surjective follows from the finiteness of the class number.

In this case one also has an expression for the special value along the lines of **Conjecture CFF** which follows from the class number formula and the functional equation. Here

$$L_\infty(\text{Spec}(K), s) = 2^{-r_2 s} \pi^{-\frac{ns}{2}} \Gamma(s/2)^{r_1} \Gamma(s)^{r_2}$$

and

$$\Lambda_K(s) = |D(K)|^{s/2} L_\infty(\text{Spec}(K), s) \zeta_K^\infty(s)$$

where  $D(K)$  is the discriminant.  $\Lambda_K(s)$  satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1 - s)$$

and has a simple pole at  $s = 0$  with residue

$$\Lambda_K^*(0) = \text{Res}_{s=0} \Lambda_K(s) = -\frac{R_K h_K}{w_K}$$

Here  $R_K$  is the classical regulator,  $h_K$  is the class number and  $w_K$  is the number of roots of unity.  $h_K$  is the order of the cokernel of the non-archimedean regulator map and  $w_K$  is the order of the kernel, so the formula is very similar to that expected in **Conjecture CFF**. In this case there is a way of defining a second integral structure on the Deligne cohomology and  $R_K$  is the cokernel of that map.

The simple pole at  $s = 0$  shows that **Conjecture C2** holds in this case as well.

### 10.1.3 Function field case

In this case the precise analogue of all the conjectures is known and can be found in the book of Rosen [Ro]. If  $K = \mathbb{F}_q(T)$ , Rosen (pg 244) shows, for example, that the completed zeta function  $\Lambda_K(s)$  has a simple pole at  $s = 0$  and

$$\Lambda_K^*(0) = -\frac{h_K}{(q-1)\log(q)}$$

which is precisely what is expected by **Conjecture C2** as  $h_K$  is the class number and is the order of the cokernel, while the kernel is the number of elements of finite order in  $K^*$  which is  $q-1$ , the cardinality of  $\mathbb{F}_q^*$ .

## 10.2 $K_2$ of curves

Here we consider the case when  $q = 2$  and  $a = 0$  so  $q - 2a = 2$ .

### 10.2.1 Local Case - Conjecture A

Here at a prime of good reduction the Deligne cohomology group is trivial, so that case is true for trivial reasons. At a prime of semistable reduction Ramakrishnan [Ra] following Bloch and Grayson [B-G] for elliptic curves defined a regulator map which coincides with ours. Bloch and Grayson showed that Conjecture A is true for a prime of split semistable reduction of an elliptic curve over  $\mathbb{Q}$ . A more precise computation of the  $v$ -adic regulator at such a prime can be found in the work of Schappacher and Scholl [S-S].

### 10.2.2 Global Case: Conjecture B

In general not much is known for curves. The usual Beilinson conjecture is known for some types of elliptic curves over  $\mathbb{Q}$  and for modular curves. The  $S$ -integral version was originally formulated by Ramakrishnan, but still remains to be proved.

### 10.2.3 Function field case

The function field we consider is  $K = \mathbb{F}_q(T)$ . This case when  $S = \{1/T\}$  is done in the works of Kondo [Ko] and Pàl [Pàl]. They define and compute the regulator map for  $K_2$  of Drinfel'd modular curves and, via the modular parametrization for the product of two elliptic curves.

## 10.3 $K_1$ of products of modular curves and elliptic curves

This is the case when  $q = 3$  and  $a = 1$ . In this case one expects a contribution from the primes of good reduction as well. From the modularity theorem, what is proved here can be used to prove a statement about products of elliptic curves over  $\mathbb{Q}$ .

### 10.3.1 Local Case - Conjecture A

Let  $E$  and  $E'$  be two elliptic curves over  $\mathbb{Q}$  and  $v$  a finite place. We are interested  $CH^2(X, 1)$  where  $X = E \times E'$ . Since they are modular, we can compute the  $L$ -function of  $H^2(X)$ . There are several cases to be considered.

First, assume  $E$  and  $E'$  are not isogenous. Suppose both have good reduction at  $v$  and  $E_v$  and  $E'_v$  are not isogenous. Then  $H_D^3(X/v, \mathbb{Q}(2))$  is two dimensional. The map  $r_{\mathcal{D}, v}$  is surjective as we have

a map  $CH^1(X, 1) \otimes CH^1(X, 0) \rightarrow CH^2(X, 1)$ . The elements in the image are called *decomposable*. Consider the cycles  $(E \times pt) \otimes 1$  and  $(pt \times E') \otimes 1$ . These map surjectively onto the Deligne cohomology so that case is taken care off. Suppose  $E_v$  and  $E'_v$  are isogenous. Then  $H_D^3(X/v, \mathbb{Q}(2))$  is 4 dimensional - the new cycles coming from the graph of the isogeny and the graph of the isogeny composed with the Frobenius. Spieß[Sp] shows how to find elements in  $CH^2(X, 1)$  whose regulator is these extra cycles.

If  $E$  and  $E'$  are isogenous, then  $E_v$  and  $E'_v$  are necessarily isogenous and so we are in case 2 above. It should be remarked that over larger residue fields, the rank of the Deligne cohomology could be as large as 6 owing in the case of supersingular reduction. However, if  $E$  is an elliptic curve over  $\mathbb{Q}$  and  $E_p$  is supersingular, the extra endomorphisms are not defined over  $\mathbb{F}_p$ .

If  $E_v$  has good reduction and  $E'_v$  does not, the  $L$ -function computation suggests that the rank of the Deligne cohomology is 2 and hence we can use the decomposable cycles to prove surjectivity. If both  $E_v$  and  $E'_v$  have semi-stable reduction, then  $L$ -function shows that the rank of the Deligne cohomology is 3. If  $E$  and  $E'$  are not isogenous, two of the elements can be obtained from the decomposable elements. The third is constructed using the modular parametrization, a construction similar to that of Milne [Mi]. Finally, if  $E$  and  $E'$  are isogenous, and  $v$  is a prime of semistable reduction, the Deligne cohomology is once again 3 dimensional and the surjectivity of the regulator map comes from the 3 decomposable elements.

The achimedean case, namely the Hodge- $\mathcal{D}$ -conjecture is due to Lewis and Chen [C-L].

### 10.3.2 Global Case - Conjecture B:

When  $S = \infty$  and  $X$  is the product of two modular curves, Beilinson [Be] proved that the regulator map is surjective and showed that the value of the  $L$ -function is equal to  $c_\infty(X, 3, 1)$  up to a rational number.

From the work of Milne, one can deduce the weak  $S$ -integral conjecture for any finite set of primes  $S$  of good or semi-stable reduction.

The exact value conjecture is still unresolved, in the case of  $E$  and  $E'$  isogenous, the special value is related to the symmetric square of  $L$ -function of a modular form and there is a lot of work on this due to Flach [Fl] and others. In [Ba-Sr], we obtain an exact formula in the non-isogenous case which may have some bearing on the conjecture.

### 10.3.3 Function Field case

The case of conjecture  $B$  with  $S = \{1/T\}$  for  $K_1$  of the product of Drinfeld modular curves was carried out by us [C-S]. We show, for example, that the special value of the  $L$ -function is  $\log(q)$  up to an element of  $\mathbb{Q}^*$ , as predicted by the conjecture, though we actually have a more precise expression for the special value.

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